

# NORDHAUS-GADDUM FOR TREewidth

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ABSTRACT. We prove that for every  $n$ -vertex graph  $G$ , the treewidth of  $G$  plus the treewidth of the complement of  $G$  is at least  $n - 2$ . This bound is tight.

Nordhaus-Gaddum-type theorems establish bounds on  $f(G) + f(\overline{G})$  for some graph parameter  $f$ , where  $\overline{G}$  is the complement of a graph  $G$ . The literature has numerous examples; see [1, 4, 5, 8, 10, 13, 14] for a few. Our main result is the following Nordhaus-Gaddum-type theorem for treewidth<sup>1</sup>, which is a graph parameter of particular importance in structural and algorithmic graph theory. Let  $\text{tw}(G)$  denote the treewidth of a graph  $G$ .

**Theorem 1.** *For every graph  $G$  with  $n$  vertices,*

$$\text{tw}(G) + \text{tw}(\overline{G}) \geq n - 2 .$$

The following lemma is the key to the proof of Theorem 1.

**Lemma 2.** *Let  $G$  be a graph with  $n$  vertices, no induced 4-cycle, and no  $k$ -clique. Then  $\text{tw}(\overline{G}) \geq n - k$ .*

*Proof.* Let  $\mathcal{B} := \{\{v, w\} : vw \in E(\overline{G})\}$ . If  $\{v, w\}$  and  $\{x, y\}$  do not touch for some  $vw, xy \in E(\overline{G})$ , then the four endpoints are distinct and  $(v, x, w, y)$  is an induced 4-cycle in  $G$ , which is a contradiction. Thus  $\mathcal{B}$  is a bramble in  $\overline{G}$ . Let  $S$  be a hitting set for  $\mathcal{B}$ . Thus no edge in  $\overline{G}$  has both endpoints in  $V(\overline{G}) \setminus S$ . Hence  $V(G) \setminus S$  is a clique in  $G$ . Therefore  $n - |S| \leq k - 1$  and  $|S| \geq n - k + 1$ . That is, the order of  $\mathcal{B}$  is at least  $n - k + 1$ . By the Treewidth Duality Theorem,  $\text{tw}(\overline{G}) \geq n - k$ , as desired.  $\square$

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<sup>1</sup>While treewidth is normally defined in terms of tree decompositions (see [3]), it can also be defined as follows. A graph  $G$  is a  $k$ -tree if  $G \cong K_{k+1}$  or  $G - v$  is a  $k$ -tree for some vertex  $v$  whose neighbours induce a  $k$ -clique. Then the *treewidth* of a graph  $G$  is the minimum integer  $k$  such that  $G$  is a spanning subgraph of a  $k$ -tree. See [2, 11] for surveys on treewidth.

Let  $G$  be a graph. Two subsets of vertices  $A$  and  $B$  in  $G$  *touch* if  $A \cap B \neq \emptyset$ , or some edge of  $G$  has one endpoint in  $A$  and the other endpoint in  $B$ . A *bramble* in  $G$  is a set of subsets of  $V(G)$  that induce connected subgraphs and pairwise touch. A set  $S$  of vertices in  $G$  is a *hitting set* of a bramble  $\mathcal{B}$  if  $S$  intersects every element of  $\mathcal{B}$ . The *order* of  $\mathcal{B}$  is the minimum size of a hitting set. Seymour and Thomas [12] proved the Treewidth Duality Theorem, which says that a graph  $G$  has treewidth at least  $k$  if and only if  $G$  contains a bramble of order at least  $k + 1$ .

*Proof of Theorem 1.* Let  $k := \text{tw}(G)$ . Let  $H$  be a  $k$ -tree that contains  $G$  as a spanning subgraph. Thus  $H$  has no induced 4-cycle (it is chordal) and has no  $(k+2)$ -clique. By Lemma 2 and since  $\overline{G} \supseteq \overline{H}$ , we have  $\text{tw}(\overline{G}) \geq \text{tw}(\overline{H}) \geq n - k - 2$ . That is,  $\text{tw}(G) + \text{tw}(\overline{G}) \geq n - 2$ .  $\square$

Lemma 2 immediately implies the following result of independent interest.

**Theorem 3.** *For every graph  $G$  with girth at least 5, we have  $\text{tw}(\overline{G}) \geq n - 3$ .*

For  $k$ -trees we have the following precise result, which proves that the bound in Theorem 1 is tight. Let  $Q_n^k$  be the  $k$ -tree consisting of a  $k$ -clique  $C$  with  $n - k$  vertices adjacent only to  $C$ .

**Theorem 4.** *For every  $k$ -tree  $G$ ,*

$$\text{tw}(G) + \text{tw}(\overline{G}) = \begin{cases} n - 1 & \text{if } G \cong Q_n^k \\ n - 2 & \text{otherwise} \end{cases}.$$

*Proof.* First suppose that  $G \cong Q_n^k$ . Then  $\overline{G}$  consists of  $K_{n-k}$  and  $k$  isolated vertices. Thus  $\text{tw}(\overline{G}) = n - k - 1$ , and  $\text{tw}(G) + \text{tw}(\overline{G}) = n - 1$ .

Now assume that  $G \not\cong Q_n^k$ . By the definition of  $k$ -tree,  $V(G)$  can be labelled  $v_1, \dots, v_n$  such that  $\{v_1, \dots, v_{k+1}\}$  is a clique, and for  $j \in \{k+2, \dots, n\}$ , the neighbourhood of  $v_j$  in  $G[\{v_1, \dots, v_{j-1}\}]$  is a  $k$ -clique  $C_j$ . If  $C_{k+2}, \dots, C_n$  are all equal then  $G \cong Q_n^k$ . Thus  $C_j \neq C_{k+2}$  for some minimum integer  $j$ . Observe that each vertex in  $C_j$  has a neighbour outside of  $C_j$ . Arbitrarily label  $C_j = \{x_1, \dots, x_{k+1}\}$ , and let  $y_i$  be a neighbour of each  $x_i$  outside of  $C_j$ .

We now describe an  $(n - k - 2)$ -tree  $H$  that contains  $\overline{G}$ . Let  $A := V(G) \setminus C_j$  be the starting  $(n - k - 1)$ -clique of  $H$ . Add each vertex  $x_i$  to  $H$  adjacent to  $A \setminus \{y_i\}$ . Observe that  $H$  is an  $(n - k - 2)$ -tree and  $\overline{G}$  is a spanning subgraph of  $H$ . Thus  $\text{tw}(\overline{G}) \leq n - k - 2$  and  $\text{tw}(G) + \text{tw}(\overline{G}) \leq n - 2$ , with equality by Theorem 1.  $\square$

In view of Theorem 1, it is natural to also consider how large  $\text{tw}(G) + \text{tw}(\overline{G})$  can be. Every  $n$ -vertex graph  $G$  satisfies  $\text{tw}(G) \leq n - 1$ , implying  $\text{tw}(G) + \text{tw}(\overline{G}) \leq 2n - 2$ . It turns out that this trivial upper bound is tight up to lower order terms. Indeed, Perarnau and Serra [9] proved that, if  $G \in \mathcal{G}(n, p)$  is a random  $n$ -vertex graph with edge probability  $p = \omega(\frac{1}{n})$  in the sense of Erdős and Rényi, then asymptotically almost surely  $\text{tw}(G) = n - o(n)$ ; see [6, 7] for related results. Setting  $p = \frac{1}{2}$ , it follows that asymptotically almost surely,  $\text{tw}(G) = n - o(n)$  and  $\text{tw}(\overline{G}) = n - o(n)$ , and hence  $\text{tw}(G) + \text{tw}(\overline{G}) = 2n - o(n)$ .

Theorems 1 and 4 can be reinterpreted as follows.

**Proposition 5.** *For all graphs  $G_1$  and  $G_2$ , the union  $G_1 \cup G_2$  contains no clique on  $\text{tw}(G_1) + \text{tw}(G_2) + 3$  vertices. Conversely, there exist graphs  $G_1$  and  $G_2$  such that  $G_1 \cup G_2$  contains a clique on  $\text{tw}(G_1) + \text{tw}(G_2) + 2$  vertices.*

*Proof.* For the first claim, we may assume that  $V(G_1) = V(G_2)$  and  $E(G_1) \cap E(G_2) = \emptyset$ . Let  $S$  be a clique in  $G_1 \cup G_2$ . Thus  $G_1[S]$  and  $G_2[S]$  are complementary. By Theorem 1,  $\text{tw}(G_1) + \text{tw}(G_2) \geq \text{tw}(G_1[S]) + \text{tw}(G_2[S]) \geq |S| - 2$ . Thus  $|S| \leq \text{tw}(G_1) + \text{tw}(G_2) + 2$  as desired. The converse claim follows from Theorem 4.  $\square$

Proposition 5 suggests studying  $G_1 \cup G_2$  further. For example, what is the maximum of  $\chi(G_1 \cup G_2)$  taken over all graphs  $G_1$  and  $G_2$  with  $\text{tw}(G_1) \leq k$  and  $\text{tw}(G_2) \leq k$ ? By Proposition 5 the answer is at least  $2k + 2$ . A minimum-degree greedy algorithm proves that  $\chi(G_1 \cup G_2) \leq 4k$ . This question is somewhat similar to Ringel's earth-moon problem which asks for the maximum chromatic number of the union of two planar graphs.

## REFERENCES

- [1] FRANCESCO BARIOLI, WAYNE BARRETT, SHAUN M. FALLAT, H. TRACY HALL, LESLIE HOGBE, AND HEIN VAN DER HOLST. On the graph complement conjecture for minimum rank. *Linear Algebra and its Applications*, 2010. doi:10.1016/j.laa.2010.12.024.
- [2] HANS L. BODLAENDER. A partial  $k$ -arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998. doi:10.1016/S0304-3975(97)00228-4.
- [3] REINHARD DIESTEL. *Graph theory*, vol. 173 of *Graduate Texts in Mathematics*. Springer, 4th edn., 2010. <http://diestel-graph-theory.com/index.html>.
- [4] ZOLTAN FÜREDI, ALEXANDR V. KOSTOCHKA, RISTE ŠKREKOVSKI, MICHAEL STIEBITZ, AND DOUGLAS B. WEST. Nordhaus-Gaddum-type theorems for decompositions into many parts. *J. Graph Theory*, 50(4):273–292, 2005. doi:10.1002/jgt.20113.
- [5] WAYNE GODDARD, MICHAEL A. HENNING, AND HENDA C. SWART. Some Nordhaus-Gaddum-type results. *J. Graph Theory*, 16(3):221–231, 1992. doi:10.1002/jgt.3190160305.
- [6] TON KLOKS AND HANS BODLAENDER. Only few graphs have bounded treewidth. Tech. Rep. RRR-CS-92-35, Utrecht University, Netherlands, 1992. <http://www.cs.uu.nl/research/techreps/repo/CS-1992/1992-35.pdf>.
- [7] CHOONGBUM LEE, JOONKYUNG LEE, AND SANG IL OUM. Rank-width of random graphs, 2010. <http://arxiv.org/abs/1001.0461>.
- [8] E. A. NORDHAUS AND JERRY W. GADDUM. On complementary graphs. *Amer. Math. Monthly*, 63:175–177, 1956.
- [9] GUILLEM PERARNAU AND ORIOL SERRA. On the tree-depth of random graphs, 2011. <http://arxiv.org/abs/1104.2132>.
- [10] BRUCE REED AND ROBIN THOMAS. Clique minors in graphs and their complements. *J. Combin. Theory Ser. B*, 78(1):81–85, 2000. doi:10.1006/jctb.1999.1930.
- [11] BRUCE A. REED. Tree width and tangles: a new connectivity measure and some applications. In *Surveys in combinatorics*, vol. 241 of *London Math. Soc. Lecture Note Ser.*, pp. 87–162. Cambridge Univ. Press, 1997.
- [12] PAUL D. SEYMOUR AND ROBIN THOMAS. Graph searching and a min-max theorem for tree-width. *J. Combin. Theory Ser. B*, 58(1):22–33, 1993. doi:10.1006/jctb.1993.1027.
- [13] MICHAEL STIEBITZ. On Hadwiger's number—a problem of the Nordhaus-Gaddum type. *Discrete Math.*, 101(1–3):307–317, 1992. doi:10.1016/0012-365X(92)90611-I.
- [14] MICHAEL STIEBITZ AND RISTE ŠKREKOVSKI. A map colour theorem for the union of graphs. *J. Combin. Theory Ser. B*, 96(1):20–37, 2006. doi:10.1016/j.jctb.2005.06.003.

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